# A simple, yet subtle "invariance" of two-body decay kinematics

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We study the two-body decay of a mother particle into a massless daughter. We further assume that the mother particle is *un*polarized and has a *generic* boost distribution in the laboratory frame. In this case, we show analytically that the laboratory frame energy distribution of the massless decay product has a *peak*, whose location is identical to the (fixed) energy of that particle in the *rest* frame of the corresponding mother particle. Given its simplicity and "invariance" under variations of the boost distribution of the mother particle, our finding should be useful for the determination of masses of mother particles. In particular, we anticipate that such a procedure will then *not* require a full reconstruction of this two-body decay chain (or for that matter, information about the rest of the event). With this eventual goal in mind, we make a proposal for extracting the peak position by fitting the data to a well-motivated analytic function describing the shape of such energy distribution. This fitting function is then tested on the theoretical prediction for top quark pair production and its decay and it is found to be quite successful in this regard. As a proof of principle of the usefulness of our observation, we apply it for measuring the mass of the top quark at the LHC, using simulated data and including experimental effects.

It is very well-known that in the *rest* frame of a mother particle undergoing a two-body decay, the energy of each of the daughter particles is fixed in terms of mother and the daughter particle masses (see, for example, Ref. [1] and the references therein). Turning this fact around, we can determine the mass of the mother particle if we can measure these rest-frame energies of the daughter particles.

However, often the mother particle is produced in the laboratory with a boost, that too with a magnitude and direction which is (a priori) not known. Moreover, the boost of mother particles produced at hadron colliders is different in each event. Such a boost *distribution* depends on the production mechanism of the particle and of the structure functions of the hadrons in the initial state of the collision and is thus a complicated function. In turn, the fact that the mother has a different boost in each event implies that when we consider the *observed* energy of the two-body decay product in the laboratory frame, we get a *distribution* in it. Thus it seems like the information that was encoded in the rest frame energy is lost and we are prevented from extracting (at least at an easily tractable level) the mass of the mother particle along the lines described above.

We show that, remarkably, if one of the daughter particles from the two-body decay is massless and the mother is unpolarized, then such is *not* the case. Specifically, in this case, we demonstrate that the distribution of the daughter particle's energy in the laboratory frame has a *peak* precisely at its corresponding rest-frame energy.

This result is interesting *per se*. Furthermore, we expect that it will lead to formulation of new methods for mass measurements. Obviously, for this purpose, we need to be able to determine the location of this peak accurately from the observed energy distribution of the mass-

less daughter. To this end, we propose and motivate an analytic function that can be used to fit the data on the energy distribution and thus extract the peak position. We show that this function is a suitable one using the top quark decay, $t \to W^- b$, as a test case, namely, it fits very well the *theory* prediction for energy spectrum of the resulting $b$-jets. Simulating a realistic experimental situation, we then show that we can extract the value of the top mass from the position of the peak in the $b$-jet energy distribution along with the well-measured mass of the $W$ boson.

Let us consider the decay of a heavy particle $B$ of mass $m_B$, i.e., $B \to A\,a$ where $a$ is a *massless* visible particle. For the subsequent arguments, the properties of the particle $A$ (other than its mass which we denote by $m_A$) are irrelevant. In the rest frame of particle $B$, the energy of the particle $a$ is simply given by:

$$E^* = \frac{m_B^2 - m_A^2}{2m_B} \, . \tag{1}$$

Here and henceforth the starred quantity denotes that it is measured in the rest frame of particle $B$, i.e., mother particle. If the mother particle (originally at rest) is boosted by a Lorentz factor $\gamma = 1/\sqrt{1 - \beta^2}$ in going to the laboratory frame, then the energy of particle $a$ seen in the laboratory frame is

$$E = E^* \gamma \left(1 + \beta \cos \theta^*\right) , \tag{2}$$

where $\theta^*$ defines the direction of emission of particle $a$ in the rest frame of $B$ with respect to the boost direction $\vec{\beta}$ of the mother $B$ in the laboratory frame. Due to our

assumption of the mother being not polarized [1] the distribution of $\cos\theta^*$ is flat. This implies that the distribution of $E$ is flat as well. More precisely, since $\cos\theta^*$ runs from $-1$ to $+1$, for any fixed $\gamma$ the shape of the distribution of $E$ is a simple "rectangle" spanning the range [2]

$$x \equiv \frac{E}{E^*} \in \left[\left(\gamma - \sqrt{\gamma^2 - 1}\right), \left(\gamma + \sqrt{\gamma^2 - 1}\right)\right], \quad (3)$$

where $x$ defines the dimensionless energy variable of the visible particle in the laboratory frame normalized by its rest-frame energy (here, we have used the relationship, $\gamma\beta = \sqrt{\gamma^2 - 1}$). A few crucial observations are in order. First, the lower (upper) bound of eq. (3) is smaller (larger) than 1, which implies that each rectangle for a fixed, but arbitrary $\gamma$ *always* contains $E^*$. In turn, this implies that, for any *distribution* of the boost of the mother particle $B$, the energy distribution of the daughter $a$ always covers the energy $E^*$. Remarkably $E^*$ is the only value of the energy to enjoy such a property, as long as the distribution of mother particle boost is non-vanishing in a small region around $\gamma = 1$. Thus, the peak of the energy distribution of the particle $a$ is unambiguously located at $E = E^*$. In fact, this argument goes through even for a *massive* daughter, provided we restrict boosts of the mother particle to $\gamma < (2\gamma^{*\,2} - 1)$, where $\gamma^*$ denotes the boost of the daughter in the rest frame of the mother. Secondly, such rectangles are *asymmetric* with respect to the point $x = 1$ (where $E = E^*$), i.e., the upper bound is farther from $x = 1$ than the lower bound. Thus, the energy distribution of the particle $a$ has a longer tail toward high energy w.r.t such a peak.

More formally, one can derive the (normalized) differential decay width in $x$ from the fact that $\cos\theta^*$ is flat as mentioned above. The result for a *fixed* $\gamma$ is

$$\left.\frac{1}{\Gamma}\frac{d\Gamma}{dx}\right|_{\text{fixed }\gamma} = \frac{\Theta\left(x - \gamma + \sqrt{\gamma^2 - 1}\right)\Theta\left(-x + \gamma + \sqrt{\gamma^2 - 1}\right)}{2\sqrt{\gamma^2 - 1}} \quad (4)$$

where $\Theta(x)$ is the usual Heaviside step function, and the two step functions here merely define the allowed range of $x$. Next, consider a probablility distribution of boosts of the mother, given by $g(\gamma)$. A given energy of daughter in laboratory frame ($x$) can actually result from a specific range of values of the mother boost ($\gamma$), as per eq. (4). So, we have to superpose these contributions, weighted by the boost distribution, giving:

$$f(x) \equiv \frac{1}{\Gamma}\frac{d\Gamma}{dx} = \int_{\frac{1}{2}\left(x+\frac{1}{x}\right)}^{\infty} d\gamma \frac{g(\gamma)}{2\sqrt{\gamma^2 - 1}}. \quad (5)$$

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[1] For example, the mother can be a scalar particle or even if it has spin, then its production process might not distinguish the various polarizations.

[2] This result is quite well-known and was used for a measurement of the $W$ mass at lepton colliders [2].

The lower end in the integral here was derived from the solution to the following equation for $\gamma$:

$$x = \gamma \pm \sqrt{\gamma^2 - 1}. \quad (6)$$

Here the positive (negative) signature is relevant for $x \geq 1$ ($x < 1$). From eq. (5) we can also compute first derivative of $f(x)$, that is

$$f'(x) = \frac{\text{sgn}(1 - x)}{2x} g\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right). \quad (7)$$

We assume that $g(\gamma)$ does not vanish for any finite non-zero value of $\gamma$. This is rather typical for particles produced at colliders. In what follows we show that this is sufficient to guarantee that there is a peak at $E^*$. We consider the two possibilities for $g(1)$. Namely, if it vanishes, then $f'(x = 1) \propto g(1) = 0$, and the distribution has its unique extremum at $E = E^*$, following from our assumption on $g(\gamma)$. If $g(1) \neq 0$, then $f'(x)$ flips its sign at $x = 1$ so that the energy distribution has a cusp at $E = E^*$. The function $f(x)$ is positive and vanishes for both $x \to 0$ and $x \to \infty$, therefore the point $E = E^*$ is necessarily the peak of the distribution in both cases.

The presence of the peak is therefore demonstrated (both heuristically and formally). In particular we have shown that the peak position does not depend on the actual boost distribution of the mother particle, as long as this distribution is non-trivial, i.e., $g(\gamma)$ is not a $\delta$-function.

As advertised at the beginning, our finding can be utilized to measure a combination of $m_A$ and $m_B$ given by $E^*$, which means that $m_B$ can be measured if $m_A$ is known, or vice versa. For this purpose, we are required to extract the location of the peak accurately from data. Clearly, having a theoretical prediction from first principles for the shape of $f(x)$ is very hard because the boost distribution $g(\gamma)$ depends on the structure functions and on the actual decay vertex of the mother particle, and hence in principle it depends on the process, the mass of the mother particle, and the collider. Nevertheless, we can still exploit some functional properties of the generic $f(x)$ which are listed below:

- $f$ is a function with an argument of $\frac{1}{2}\left(x + \frac{1}{x}\right)$, i.e., it is even under $x \leftrightarrow \frac{1}{x}$,

- $f$ is maximized at $x = 1$,

- $f$ vanishes as $x$ approaches 0 or $\infty$,

- $f$ becomes a $\delta$-function in some limiting configuration.

The first property follows from eq. (5), and the second from eq. (7) and the argument thereafter. The third one is also manifest from eq. (5) since those two limits lead to a trivial definite integral from $\infty$ to $\infty$. Finally, the last one reflects the fact that no boost returns a fixed value of energy given in eq. (1), i.e., $\delta$-functionlike distribution.
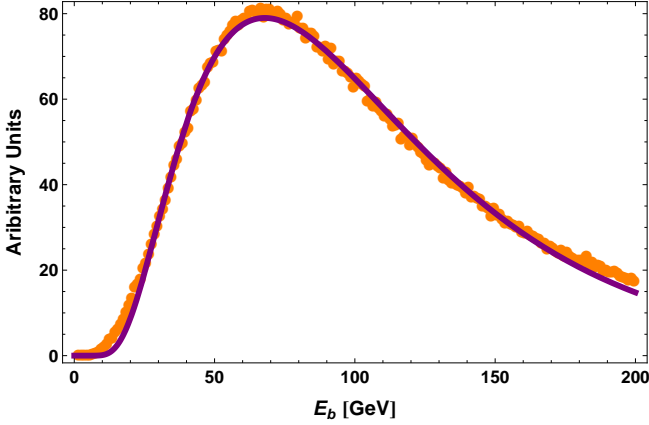
FIG. 1: The orange dots are the theory prediction for $d\sigma/dE_b$ in the process eq. (10) computed with `MadGraph5` at LHC with $\sqrt{s}$=7 TeV. The purple line is the best fit of our *ansatz* eq. (8).



FIG. 2: An instance of the result of the fit on the energy distribution of the $b$-jets in a pseudo-experiment. For the fit we binned data in bins of 4 GeV. Only the blue data points are used in the fit, which correspond to using only the part of the spectrum from 30 to 150 GeV.

Being aware of the constraints given above, we propose the following "simple" function as an *ansatz* for $f(x)$:

$$f(x) = K_1^{-1}(p) \exp\left[-\frac{p}{2}\left(x + \frac{1}{x}\right)\right], \qquad (8)$$

where $p$ is a parameter which encodes the width of the peak and the normalization factor $K_1(p)$ is a modified Bessel function of the second kind of order 1. One can easily prove that the proposed ansatz can be reduced to a $\delta$-function for any sufficiently large $p$ using the asymptotic behavior of $K_1(p)$ such that

$$K_1(p) \xrightarrow{p\to\infty} \sim \frac{e^{-p}}{\sqrt{p}}\left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right). \qquad (9)$$

Such a behaviour is required in order to reproduce the limit of no boost of the mother. Finally, we can show that the above ansatz does not have a cusp (at $E^*$) so that it is more suitable for the case of $g(1) = 0$ such as pair-production of mothers [3].

In order to test the goodness of the *ansatz* given in eq. (8) we use it to fit a *theoretical* prediction for the distribution of $b$-jet energy in top decay. The bottom quark is not massless; it is nonetheless highly boosted in the rest frame of top quark, namely, $\gamma^* \approx 15$. Based on our earlier discussion of the massive case, our argument for the peak in $b$-jet energy being at $E^*$ is invalidated for boosts of the top quark which are so large ($\gamma \gtrsim 500$) as to have a negligible probability. Hence, we expect the peak to be very close to $E^*$. Similarly we expect that the first of the functional properties of the energy spectrum eq. (5) will be only negligibly violated by the non-zero
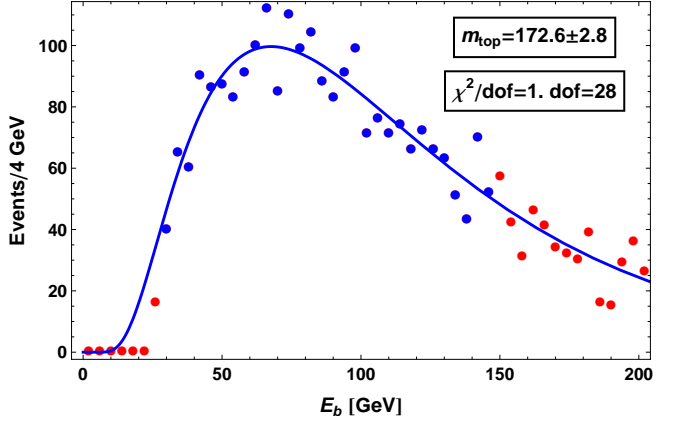
mass of the bottom quark. This justifies the use of the *ansatz* eq. (8) to fit $b$-jet energy spectrum.

Specifically, we study a sample of fully leptonic top decays from the process

$$pp \to t\bar{t} \to b\bar{b}\mu^- e^+ \nu_e \bar{\nu}_\mu \qquad (10)$$

at the Large Hardron Collider (LHC) with 7 TeV center-of-mass energy. To compute the theory prediction for the process eq. (10) we employ `MadGraph5` 1.4.2 [3] as a matrix element generator. We also take $m_{\text{top}}$ of 173 GeV and the patron distribution functions (PDFs) `CTEQ6L1` [4] evaluated with a renormalization and factorization scale varied depending on the kinematics of each event according to the default of `MadGraph5`.

The result of the associated fit is exhibited in FIG. 1 which shows a very good agreement between the theory prediction from `MadGraph5` and the fitting function. To quantify the goodness of the *ansatz* with an objective measure we compute both the Kolmogorov-Smirnov (KS) [5] and the $\chi^2$ value. The latter is computed taking bin counts for a luminosity of $5fb^{-1}$ at LHC with $\sqrt{s} = 7$ TeV assuming that the error on each bin count is gaussian. The result is $\chi^2 = 39.3$ for 198 degrees of freedom while the KS test statistic is 0.012. Neither the $\chi^2$ nor the KS test has any particular probabilistic meaning attached to it, in fact they just serve the purpose of quantifying in an objective manner the agreement between the theory prediction and the *ansatz*. The $\chi^2$ should be sensitive to deviations in regions of the distribution where there are more events, i.e., at the peak, while the KS test should be sensitive to the overall shape. Both tests indicate that our *ansatz* gives a very good fit to the theory curve. Although not shown in the figure, the agreement is very good also in the high-energy tail. We have investigated the sensitivity of this result to the choice of the PDFs by repeating the same fit on the theory prediction obtained using the `MRST2002NLO` PDFs set of Ref. [6]. We

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[3] We defer the study of the other case of $g(1) \neq 0$, for example, the single production of the mother, for future work.

observe negligible differences w.r.t. the result obtained with `CTEQ6L1`.

So far, we have found that the *ansatz* in eq. (8) is very good at reproducing the theory prediction. In fact, this success suggests that the *ansatz* may be used to measure the combination of masses in eq. (1) from *experimental* data. In order to investigate this possibility, we go back to the example of the top quark, namely, we would like to use the fitting function in order to extract the peak of the observed energy distribution of the $b$-jet and measure the top quark mass by plugging this value and the well-known mass of the $W$ boson into eq. (1). Before getting into details, we would like to mention that, we do not necessarily aim at getting a result for the value of $m_{\text{top}}$ that is competitive with the current measurements. Rather we aim at finding what is the sensitivity of our method for measuring top quark mass in a realistic setup. In fact for a fair comparison it should remarked that the current measurements of $m_{top}$ rely on rather complicated tools and often advocate templates for the distributions that require a detailed knowledge of the underlying dynamics of the top quark decay, i.e., the matrix element of the process. On the contrary, our method is extremely simple: it is based on pure kinematics and does not rely at all on detailed knowledge of the above-mentioned dynamics (as long as the top quark is produced unpolarized). As such we can regard our study of the mass measurement of the top as a proof of principle that our method can be used to measure the mass of heavy particles, in particular, new physics particles.

In lieu of actual experimental data we use a sample of Monte Carlo (MC) *simulated* collision events. Namely, we further process the event sample generated by `MadGraph5` 1.4.2 to include the effects of showering and hadronization as described in `PYTHIA` 6.4 [7] with detector response simulated by `Delphes` 1.9 [8] and jets made with `FastJet` [9, 10] using the *anti-$k_T$* algorithm [11] with the parameter choice $R = 0.4$. Furthermore, we impose cuts on the final state of the process eq. (10) following the selections of Ref. [12] for the $e\mu$ final state. We consider an ensemble of 100 pseudo-experiments, each of which is equivalent to $5\,fb^{-1}$ of data from the LHC at $\sqrt{s} = 7$ TeV. For each pseudo-experiment we perform a fit with our template eq. (8). From the extracted value of the peak of the distribution we get a measurement of $m_{\text{top}}$. The distribution over the 100 pseudo-experiments of $m_{\text{top}}$ and its $1\sigma$ error are symmetric around the central values, and do not show special features. For a bin size of 4 GeV the average best-fit $m_{\text{top}}$ and the $1\sigma$ error resulting from the fit are

$$\langle m_{\text{top}} \rangle = 173.1 \pm 2.5 \text{ GeV} \qquad (11)$$

with a median of the reduced $\chi^2$ of our fit equal to 1.1. We have checked that the result of the fit is stable under changes of the range of energies taken in the fit and changes of the bin size. For illustration purpose we show the outcome of one of the pseudo-experiments in FIG. 2.

The obtained value of $m_{\text{top}}$ is rather good: the error is small and the central value is compatible with the value from the standard methods (within the $1\sigma$ error of the latter). Furthermore, the obtained $\chi^2$ is good. All this tells that the usefulness of our function eq. (8) is not spoiled by the selection criteria for top quark decay events nor by detector effects. Even though our assessment of the power of the technique does not take the background and the entire realm of detector effects into consideration, we regard it as rather encouraging for the determination of the masses of heavy particles at colliders based only on the kinematics of the decay.

In conclusion, we have shown that for the two-body decay of an unpolarized boosted mother particle, the energy spectrum of a massless daughter in the laboratory frame encodes in a rather simple manner information about the masses involved in the decay. Specifically, we showed that, for a generic distribution of the boost of the mother particle, the location of the peak in the energy distribution of the visible daughter is identical to its rest-frame energy.

We also found a simple function that seems to reproduce rather well the energy spectrum of the massless daughter. In particular, we assessed the goodness of this function by comparing it against the theory prediction for the energy distribution of $b$-jets in the leptonic decay of top quark pairs.

Furthermore we have studied how well our simple function can be used to reproduce the distribution from MC simulated events, where the effect of soft QCD radiation and detector effects have been included. We found that our template function can be successfully used even in the presence of such effects. In fact, our finding can be used to extract the top mass in the LHC data with a rather good accuracy, well below 10%. Note that our method exhibits a remarkable twist in the mass measurement paradigm. Instead of using longitudinally or fully Lorentz invariant quantities for this purpose, we extracted the mass of the top quark from a Lorentz-variant observable, i.e., energy of $b$-jet. The crucial point is that even though the distribution of this quantity is dependent on the possible boosts of the mother particle, the location of the peak in it is "invariant".

While currently our technique might not compete with the standard (well-developed over more than a decade) methods to measure $m_{\text{top}}$ we remark that this observation can be used to get a rather precise mass measurement of *any* heavy particle, in particular, new ones. The merit of our method is that it does not rely on any measurement of the other particle of the two-body decay so that we can extract some information about masses even if the latter is invisible, a case that would be not tractable with, for example, measurement of invariant mass only. It is also clear that our method and the traditional techniques for mass measurement are sensitive to different kind of detector effects. In general, we envisage that there will be a large degree of complementarity of our

method with more traditional ones.

Finally, we emphasize that the proposed technique, despite being based on a fitting function, relies only on the minimal assumptions of absence of polarization and and the presence of a non-trivial boost distribution of the mother particle, i.e., it does not require any other prior knowledge about the underlying physics model governing the decay of the particle whose mass we want to measure. This suggests that our method will be especially suitable for the mass measurement of new particles to be discovered at the LHC, where we (a priori) would not know such details.

In forthcoming work we shall describe a number of applications of our finding here about the distributions of the energy of a daughter particle from a two-body decay. In particular, we shall present results on the measurement of the masses of *specific* new physics particles [13].

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